The vicissitudes of the variance estimation of the Gini index

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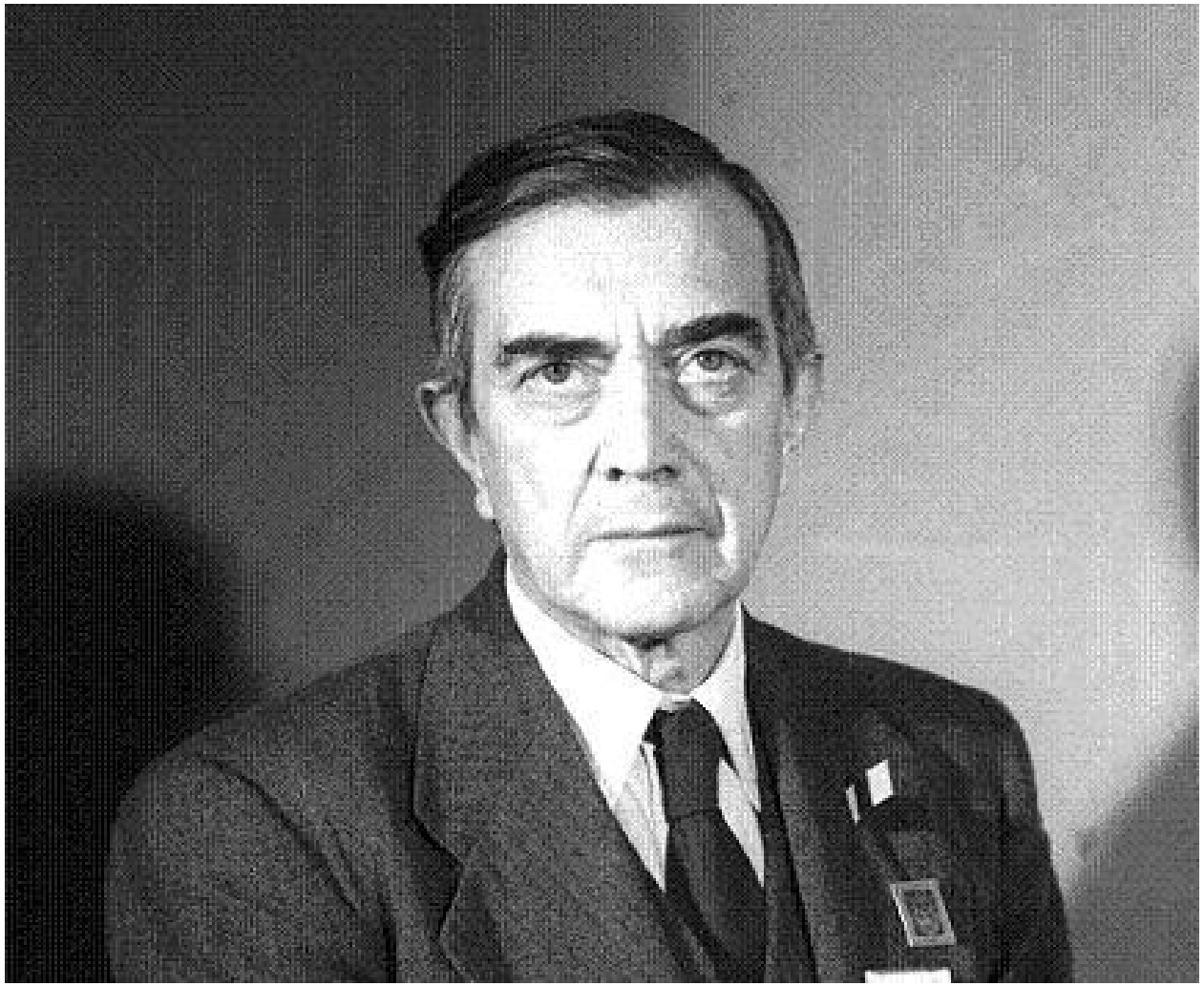
Corrado Gini



Gini coefficient

Gini coefficient

- Gini, C. (1912). *Variabilità e Mutabilità*. Bologna: Tipografia di Paolo Cuppin.
- Gini, C. (1914). Sulla misura della concertrazione e della variabilità dei caratteri. *Atti del Reale Istituto Veneto di Scienze, Lettere e Arti, LXXIII* **73**, 1203–1248.
- Gini, C. (1921). Measurement of inequality and incomes. *The Economic Journal* **31**, 124–126.

Lorenz curve

- Lorenz, M. O. (1905). Methods of measuring the concentration of wealth. *Publications of the American Statistical Association* **9**, 209–219.
- $f(y)$: probability density function of a positive continue random variable X that represents the income.
- $F(y)$: cumulative distribution function.
- Lorenz curve:

$$L(\alpha) = \frac{\int_0^{F^{-1}(\alpha)} yf(y)dy}{\int_0^{\infty} yf(y)dy} = \frac{1}{\mu}\int_0^{\alpha} F^{-1}(u)du,$$

where $F^{-1}(.)$ is the inverse function of $F(.)$ and

$$\mu = \int_0^{\infty} yf(y)dy.$$

Lorenz curve

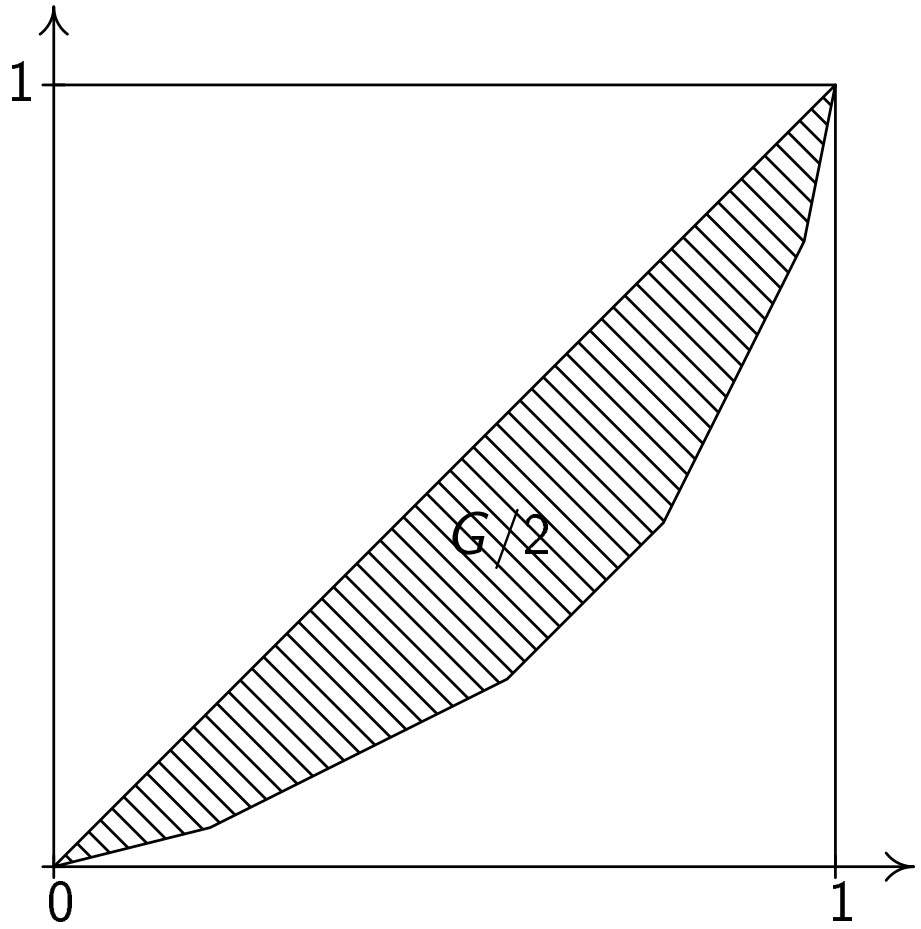


Figure: Lorenz curve and area associated to the Gini Coefficient, "The curve is a graph showing the proportion of overall income or wealth assumed by the bottom x% of the people" Wikipedia

Gini index

- Gini index from the Lorenz curve:

$$\begin{aligned} G &= 2\int_0^1 [\alpha - L(\alpha)]\, d\alpha = 1 - 2\int_0^1 L(\alpha) d\alpha \\ &= \frac{2}{\mu}\int_0^\infty F(y) f(y) d(y) - 1. \end{aligned}$$

- Discrete version: $y_{(i)}, i = 1, \ldots, n$ ordered (by increasing order) independent realization,

$$\widehat{G} = \frac{2\sum_{i=1}^n i y_{(i)}}{n\sum_{i=1}^n y_{(i)}} - \frac{n+1}{n}.$$

- Controversy $\widehat{\widetilde{G}} = n\widehat{G}/(n-1)$ rather than $\widehat{G}$.

Gini index in sampling

- In a finite population U of size N with $Y = \sum_{k \in U} y_k$.

$$G = \frac{2}{YN} \sum_{k \in U} k y_{(k)} - \frac{N+1}{N} = \frac{\sum_{k \in U} \sum_{\ell \in U} |y_k - y_\ell|}{2NY}.$$

- From a sample of $S \subset U$ with sampling weights w_k (we can have $w_k = 1/\pi_k$)

$$\begin{aligned} \widehat{G} &= \frac{2}{\widehat{N}\widehat{Y}} \sum_{k \in S} w_k \widehat{N}_k y_k - \left(1 + \frac{1}{\widehat{N}\widehat{Y}} \sum_{k \in S} w_k^2 y_k\right) \\ &= \frac{\sum_{k \in S} \sum_{\ell \in S} w_k w_\ell |y_k - y_\ell|}{2\widehat{N}\widehat{Y}}. \end{aligned}$$

with $\widehat{Y} = \sum_{k \in S} w_k y_k$, $\widehat{N} = \sum_{k \in S} w_k$, $\widehat{N}_k = \sum_{\ell \in S} w_k \mathbb{1}\{y_\ell \leq y_k\}$.

Gini index in sampling

- Sandström, A., Wretman, J. H. & Waldén, B. (1985). Variance estimators of the Gini coefficient: Simple random sampling. *Metron* **43**, 41–70.
- Sandström, A., Wretman, J. H. & Waldén, B. (1988). Variance estimators of the Gini coefficient: probability sampling. *Journal of Business and Economic Statistics,* **6**, 113–119.
- They consider that the Gini coefficient is a ratio of two totals ($iy_{(i)}$ and $y_{(i)}$)

$$\widehat{G} = \frac{2\sum_{i=1}^{n} iy_{(i)}}{n\sum_{i=1}^{n} y_{(i)}} - \frac{n+1}{n}.$$

- They compute the variance of a ratio.
- Overestimation by 10 compared to simulations.

Gini index in sampling

- What is the problem:
- We can reformulate on the sample:

$$\widehat{G} = \frac{2\sum_{i=1}^{n} i y_{(i)}}{n\sum_{i=1}^{n} y_{(i)}} - \frac{n+1}{n} = \frac{2\sum_{i=1}^{n} \mathrm{rank}(y_i) y_i}{n\sum_{i=1}^{n} y_i} - \frac{n+1}{n}.$$

- $\mathrm{rank}(y_i)$ is a random variable and depends on all the y_i.
- $\mathrm{rank}(y_i) = \sum_{k \in S} \mathbb{1}(y_k \leq y_i)$.
- If one selects large (resp. small) y_i values the ranks will be smaller (larger).
- There is a kind of compensation in the product $\mathrm{rank}(y_i) y_i$.
- If one does not consider that the rank is random, one overestimates the variance.

Gini index in sampling

- It is better to write:

$$\widehat{G} = \frac{2\sum_{i\in S}\text{rank}(y_i)y_i}{n\sum_{i\in S}^{n} y_i} - \frac{n+1}{n}.$$

with $\text{rank}(y_i) = \sum_{k\in S} \mathbb{1}(y_k \leq y_i)$.

- Or to write:

$$\widehat{G} = \frac{2\sum_{i\in U}\text{rank}(y_i)y_i a_i}{n\sum_{i\in U}^{n} y_i a_i} - \frac{n+1}{n}.$$

with $\text{rank}(y_i) = \sum_{k\in U} \mathbb{1}(y_k \leq y_i)a_k$ and $a_i = 1$ if $i \in S$ and 0 otherwise.

- a_i is correlated with $\text{rank}(y_i)$.

Influence function

- Influence function. Tool for robustness.
  - Hampel, F. R. (1974). The influence curve and its role in robust estimation. *Journal of the American Statistical Association* **69**, 383–393
  - Hampel, F. R., Ronchetti, E., Rousseeuw, P. J. & Stahel, W. A. (1985). *Robust Statistics: The Approach Based on the Influence Function*. New York: Wiley
- Influence function of the Gini coefficient
  - Glasser, G. (1962). Variance formulas for the mean difference and coefficient of concentration. *Journal of the American Statistical Association* **57**, 648–654
  - Monti, A. C. (1991). The study of the Gini concentration ratio by means of the influence function. *Statistica* **51**, 561–577
  - Cowell, F. A. & Victoria-Feser, M.-P. (1996). Robustness properties of inequality measures. *Econometrica* **64**, 77–101
  - Cowell, F. A. & Victoria-Feser, M.-P. (2003). Distribution-free inference for welfare indices under complete and incomplete information. *Journal of Economic Inequality* **1**, 191–219

Influence function

The influence function of the Gini coefficient:

$$z_k = \frac{1}{NY}\left[2N_k(y_k - \overline{Y}_k) + Y - Ny_k - G(Y + y_k N)\right],$$

where

$$\overline{Y}_k = \frac{\sum_{\ell \in U} y_\ell \mathbb{1}(y_\ell \leq y_k)}{N_k}$$

and

$$N_k = \sum_{\ell \in U} \mathbb{1}(y_\ell \leq y_k)$$

is the rank of k in the population.

Linearization

- The main idea is to approximate the estimator

$$\widehat{G} - G \approx \sum_{k \in S} w_k z_k - \sum_{k \in U} z_k.$$

- Next $\mathrm{var}(\widehat{G}) \approx \mathrm{var}\left(\sum_{k \in S} w_k z_k\right)$
- Use of the influence function to linearize.
- Deville, J.-C. (1996). Estimation de la variance du coefficient de Gini estimé par sondage. In *Actes des Journées de Méthodologie Statistique*, vol. 69-70-71. Paris: Insee-Méthodes.
- Deville, J.-C. (1999). Variance estimation for complex statistics and estimators: linearization and residual techniques. *Survey Methodology* **25**, 193–204

Regression approach 1

- Lerman, R. I. & Yitzhaki, S. (1984). A note on the calculation and interpretation of the Gini index. *Economic Letters* **15**, 363–368.
- The coefficient

$$\widehat{\beta} = \frac{1}{n} \sum_{i=1}^{n} \left(i - \frac{n(n+1)}{2} \right) y_{(i)}$$

- is the covariance between the y_i and the centered ranks.
- The Gini index depends on this coefficient.

Regression approach 2

- Giles, D. E. A. (2004). Calculating a standard error for the Gini coefficient: some further results. *Oxford Bulletin of Economics and Statistics* **66**, 425–433.
- The coefficient

$$\widehat{\theta} = \frac{\sum_{i=1}^{n} i y_{(i)}}{\sum_{i=1}^{n} y_{(i)}}$$

- is the regression coefficient of the regression $i = \theta + \nu_i$ where $\mathrm{var}(\nu_i) \propto 1/y_i$.
- Compute the variance of a regression estimator.
- Same error $y_{(i)}$ and the ranks are random, which is not the case in regression. The order statistics $y_{(i)}$ are not independent.
- Same error.

Linearization

- Davidson, R. (2009). Reliable inference for the Gini index. *Journal of Econometrics* **150**, 30–40. Republication of the linearized variable without mentioning to Deville, Monti, Victoria-Feser and Cowell.
- Langel, M. & Tillé, Y. (2013). Variance estimation of the Gini index: Revisiting a result several times published. *Journal of the Royal Statistical Society* **A176**, 521–540.

Graf approach

- Graf, M. (2010). Use of survey weights for the analysis of compositional data. Working paper, Swiss federal statistical office
- Write the estimator with the a_k

$$\widehat{G} = \frac{\sum_{k \in U} \sum_{\ell \in U} a_k a_\ell w_k w_\ell |y_k - y_\ell|}{2 \sum_{k \in U} w_k a_k \quad \sum_{k \in U} w_k a_k y_k}.$$

- $a_k = 1$ if $k \in S$ and 0 otherwise.
- Compute $z_j = \dfrac{\partial \widehat{G}}{\partial a_j}$.

$$\hat{z}_j = \frac{\partial \widehat{G}}{\partial a_j} = \frac{1}{\widehat{N}\widehat{Y}} \left[2\widehat{N}_j (y_j - \widehat{\overline{Y}}_j) + \widehat{Y} - \widehat{N} y_j - \widehat{G}(\widehat{Y} + y_j \widehat{N}) \right],$$

Conclusion

- The variance must be computed in function of the sources of randomness.
- We always repeat the same errors.
- No cross-reading between economists and statisticians.
- The problem is solved for a long time!
- There are still people using the variance based on the regression.

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